Pergamon

J. Appl. Maths Mechs, Vol. 58, No. 3, pp. 569–572, 1994

0021–8928/94 $24.00+0.00

0021-8928(94)00083-2

THE USE OF THE METHOD OF INTEGRATED PHOTOELASTICITY TO DETERMINE THE STRESSES IN A CUBIC SINGLE CRYSTAL†

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(*Received* 7 *October* 1992)

The mathematical principles of a method of determining the residual stresses in long cubic single crystals in a state of plane elastic deformation is given. The values of the absolute and relative ray path differences are the initial information. The differences are measured by a tomographic method in a plane which is orthogonal to the axis of the sample.

The use of integrated photoelasticity (IPE) methods is a promising direction for the development of non-destructive techniques for determining the stresses in single crystals [1]. These methods are based on the combined solution of the problem of the optical tomography of the stress field tensor and the problem in the theory of elasticity which arises here [2, 3]. Compared with isotropic objects, the solution of these two problems becomes much more difficult in the case of single crystals due to the occurrence of natural anisotropy [4]. In the first problem, this is associated with the non-coincidence of the quasiprincipal directions of the stress tensor and the permittivity tensor [5]. In the second problem, it is associated with the fact that, even in a state where there is a gradient-free stress distribution along the axes of a prism, the torsional and plane elastic strain equations are not separable. The effect of anisotropy is reduced on illumination in a plane of elastic symmetry [6]. The potential possibilities of the method have been demonstrated in this case on actual examples on the reconstruction of the axially symmetric stresses in cylinders [6, 7].

In the development of the IPE method [8] below, the problem of determining the residual stresses in a long cubic single crystal is considered. The axis of this crystal coincides with the crystallographic [001] direction. It is assumed that the sample is prepared under steady technological conditions so that the stress distribution in the central part of the sample corresponds to a state of plane elastic strain $\varepsilon_{xz}=\varepsilon_{yz}=\varepsilon_{zz}=0$. Usually, the influence of end effects becomes insignificant at distances of one to two diameters and the change in the stresses along the length of the sample is due solely to instability in the crystal growth process.

The problem, in this formulation, has only been considered earlier in the case of an axially symmetric stress distribution [6, 7] and subject to the condition that the residual deformation tensor is spherical and can be described by a single parameter, a fictitious temperature [9]. Below, it is proposed to measure the absolute as well as the relative path differences in order to avoid these constraints. The problems involved in measuring this parameter and its use in reconstructing the stresses in isotropic bodies have repeatedly been discussed in the literature [10–12].

1. We will introduce an orthogonal system of coordinates x, y, z and direct the axes of this system along the [100], [010] and [001] crystallographic directions so that the direction of the z-axis coincides with the generatrix of the single crystal sample (Fig. 1). Additionally, we will introduce an orthogonal system of

†*Prikl. Mat. Mekh.* Vol. 58, No. 3, pp. 186–188, 1994.

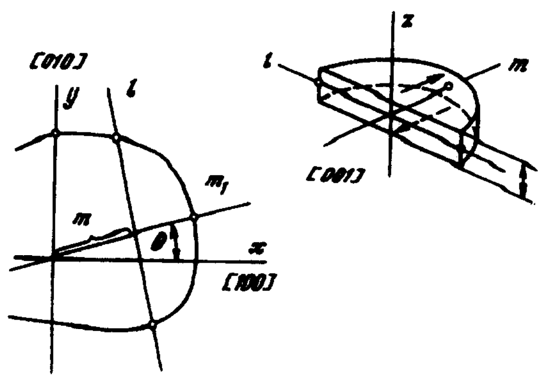

Fig. 1.

coordinates l, m, z which is rotated with respect to the initial system by an angle θ so that the direction of illumination coincides with the direction of the l-axis. With illumination in the plane $z = \text{const}$, there is no rotation of the quasiprincipal directions of the permittivity tensor. The problem of optical tomography is simplified. The refractive index of rays polarized along the plane of illumination $n_m = n + \pi_{ml}\sigma_{nm} + \pi_{m2}\sigma_{ll} + \pi_{m3}\sigma_{zz} + \pi_{m6}\sigma_{ml}$ and of rays polarized orthogonally to it $n_n = n + \pi_{zl}\sigma_{mm} + \pi_{z3}\sigma_{zz} + \pi_{z6}\sigma_{ml}$ determine the absolute $K(m, \theta) = \frac{1}{2}\int(n_m - n_z)dl$ and relative $\sigma(m, \theta) = \int(n_m - n_z)dl = A(m, \theta, z)$ path differences measured in the beam. Here n is the refractive index of the crystal in a stress-free state and π_{ik} is the fourth-order elasto-optic tensor [13]. The medium is assumed to be homogeneous and these coefficients therefore depend solely on the angle θ and can be taken outside the integral sign. The expressions for the absolute and relative path differences become

$$K(m,\theta) = n\Delta l + f_1(m,\theta), \quad \delta(m,\theta) = f_2(m,\theta) \tag{1.1}$$

$$f_k(m,\theta) = \pi_{k1}\int\sigma_{mm}dl + \pi_{k2}\int\sigma_{ll}dl + \pi_{k3}\int\sigma_{zz}dl + \pi_{k6}\int\sigma_{ml}dl, \quad k = 1,2$$

$$\pi_{1i} = \pi_{mi} + \pi_{zi}, \quad \pi_{2i} = \pi_{mi} - \pi_{zi}$$

and are expressed only in terms of radial integrals of the stresses.

Here Δl is the length of the ray in the crystal. We shall further simplify the problem on the basis of the conditions of macrostatics (integration is carried out over S, the cross-sectional area of the prism)

$$\int\sigma_{zz}ds = 0 \tag{1.2}$$

and the equilibrium equations. The latter are satisfied by introducing the Airy function F (δ_{ij} is the Kronecker delta)

$$\sigma_{ij} = \delta_{ij}\Delta F - \partial^2 F / \partial i \partial j$$

It follows from the no-load condition of the lateral surface that the Airy function F itself and its normal derivative $\delta F / \delta n$ on the contour of the domain are equal to zero [14] and the radial integrals

$$\int\sigma_{mm}dl = \int\frac{\partial^2 F}{\partial l^2}dl = \int\sigma_{lm}dl = -\int\frac{\partial^2 F}{\partial l \partial m}dl = 0$$

and expressions (1.1) therefore contain only integrals of the stress tensor components σ_{zz}.

The refractive index of the crystal $\int\sigma_{ll}ds = \int(\int\sigma_{ll}dm)dl = 0$ is determined from the macrostatics condition (1.2) and the relationships $n = \int K(m, \theta)dm / S$ and, consequently, the Radon transformations of σ_{zz} and the invariant $\sigma_{mm} + \sigma_{ll} = \Delta F$

$$\int\sigma_{ll}dl = \int\frac{\partial^2 F}{\partial m^2}dl = \int\Delta F dl$$

can be determined separately from measurements of the parameters $K(m, \theta)$ and $\delta(m, \theta)$.

Hence, as in the case of an isotropic medium [8], the values of σ_{zz} and

$$\Delta F = f(x, y) \tag{1.3}$$

are found from the Radon transform of the linear combination of radial integrals (1.1) while the transverse components of the stress tensor are found from the solution of Poisson's equation (1.3) with overdetermined conditions $F = \partial F / \partial n = 0$ on the contour. The overdetermination of the boundary conditions can be used to reduce the errors when reconstructing the stresses [8].

2. In the case of a small rotation of the quasiprincipal directions along the radial line (a smooth change in the stresses along the axis of the sample), the radial integrals [4, 15, 16] can be determined from the measured characteristic phase difference δ and the isocline parameter ψ

$$\delta\cos 2\psi = \int (n_m - n_z) dl = A(m,\theta,z), \quad \delta\sin 2\psi = \int n_{mz} dl = C\int \sigma_{mz} dl = CH(m,\theta,z) \tag{2.1}$$

We then immediately obtain the integral relationship [3]

$$\int \frac{\partial\sigma_{zz}}{dl} = -\frac{\partial}{\partial m} H(m,\theta,z) \tag{2.2}$$

from the equilibrium condition for an element S in the z-direction, which reduces the problem of finding $\partial\sigma_{zz} / \partial z$ to the standard procedure for the inversion of a Radon transform.

The tensor components are found layer-by-layer, moving along the sample from a section with a known stress distribution. Using this technique, the values of the integral (2.2) enable one to recover σ_{zz} in the whole sample and to eliminate this element of the stress tensor from the integrals K and A. Consequently, by using a linear combination of the radial integrals $K(m, \theta, z)$, $A(m, \theta, z)$ and the condition for the equilibrium of an element S in the direction m [3]

$$\int \sigma_{mm} dl = \frac{\partial}{\partial z} \int_m^{m_1} H(k,\theta,z) dk \tag{2.3}$$

the determination of the components σ_{xx}, σ_{xy}, σ_{yy} can be reduced to the standard procedure for the inversion of a Radon transform. The upper limit m_l in the integral is the value of the projection of one of the end points of the contour on the m axis. We recall that the above-mentioned components of the stress tensor are a linear combination of the components σ_{ll} σ_{lm}, σ_{mm}, which appear in the radial integrals (2.1) and (2.3).

The increments in the tangential components after this are determined from the equilibrium equations

$$-\partial\sigma_{iz} / \partial z = \partial\sigma_{ij} / \partial j + \partial\sigma_{ii} / \partial i; \quad i,j = x,y$$

This technique therefore enables one to determine completely the stresses in a cubic single crystal not only in the case of plane deformation but also when there is a small change in the stresses along the axis of the sample.

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Translated by E.L.S.